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Complex Ray Configurations in an Ionosphere Composed of Spherical Shells

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13. ABSTRACT

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ABSTRACT

In this work some complex ray configurations are treated using a simple ionospheric model, namely, spheres concentric with a spherical earth. Other simplifying assumptions are also made. Methods for finding takeoff angle of ray and length of ray path are given.

PROBLEM STATUS

This is an interim report on one phase of the NRL problem.

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Complex Ray Configurations in an Ionosphere Composed of Spherical Shells

INTRODUCTION

Oftentimes a ray configuration is so complex that only the simplest of ionospheric models can be employed successfully in computing length of ray path and delay time. Of course when such simple models are used, the computed delay time is only an approximation. However, in many instances an approximation is all that is needed.

In this work some complex ray configurations are treated using a simple ionospheric model, namely, spheres concentric with a spherical earth. Other simplifying assumptions are also made.

Ray Configurations for an Ionosphere Composed of Spherical Shells

I. Assumptions

The following assumptions are made:

1. The earth is a sphere of radius r.
2. The E_i layer of the ionosphere is a sphere, concentric with the earth, of height h_i above the earth's surface.
3. All reflections are such that the angle of incidence is equal to the angle of reflection.
4. The ray path is composed of straight line segments.

II. Ray Configuration

The ray configuration, earth-E_i layer-earth, will be called a hop to the E_i layer.

The following ray configuration is considered:

n_1 hops to the E_1 layer,
n_2 hops to the E_2 layer,
. . .
n_p hops to the E_p layer, where $p = 1, 2, 3, \ldots$; the n_i are non-negative integers.

In most problems p does not exceed 2, and the ionospheric layers considered are the E, F_1, and F_2 layers. The notation E_i is used simply for mathematical convenience.

III. Statement of Problem

Compute the total distance that the ray travels for the above configuration.

IV. Mathematical Model

Let ξ_i be the angle made by the ray on reflecting off the E_i layer (Fig. 1).

Let T and L be, respectively, the points of takeoff and landing of the ray. (T and L being on the earth's surface).

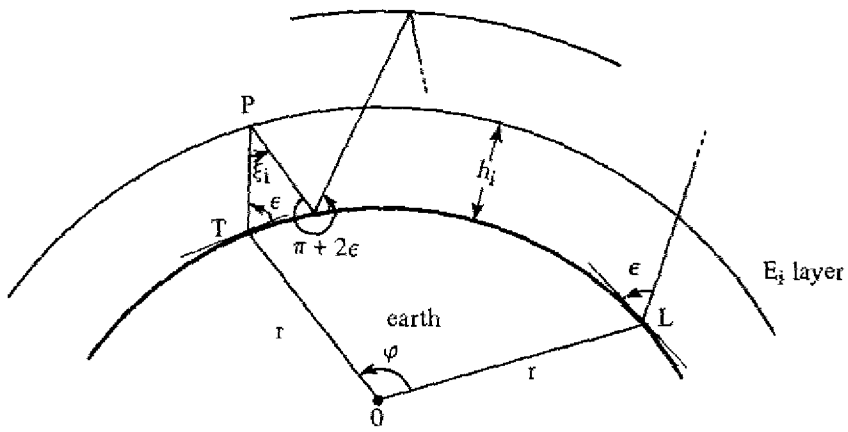

Fig. 1 — Earth – E_i layer – earth reflections.

Let φ be the angle between T and L.
Let ϵ be the takeoff angle of the ray.
Let d be the total length that the ray travels in going from T to L.

It is agreed that all angles are measured counterclockwise.

Now consider the polygon formed by the ray configuration and the radii of the earth (Fig. 1).

Since the sum of the interior angles of a polygon is equal to $(n - 2)\pi$, where n is the number of vertices of the polygon,

$$\varphi + 2\left(\frac{\pi}{2} + \epsilon\right) + \sum_{i=1}^{P} n_i \xi_i + \left(-1 + \sum_{i=1}^{P} n_i\right)(\pi + 2\epsilon) = 2\pi \sum_{i=1}^{P} n_i.$$

Solving this equation for ϵ gives

$$\epsilon = \frac{1}{2}\left[\left(\pi - \frac{\varphi}{N}\right) - \frac{1}{N}\sum_{i=1}^{P} n_i \xi_i\right], \quad \text{where } N = \sum_{i=1}^{P} n_i.$$

Applying the law of sines to the (typical) triangle OTP (Fig. 1) gives

$$\xi_i = 2 \arcsin (K_i \cos \epsilon) \quad \text{where } K_i = \frac{r}{r + h_i}.$$

Hence

$$\epsilon = \frac{1}{2}\left(\pi - \frac{\varphi}{N}\right) - \frac{1}{N}\sum_{i=1}^{P} n_i \arcsin (K_i \cos \epsilon). \tag{1}$$

If only one layer, whose height is h, and n hops to this layer are being considered, equation (1) reduces to

$$\epsilon = \frac{1}{2}\left(\pi - \frac{\varphi}{n}\right) - \arcsin\left(\frac{r}{r+h} \cos \epsilon\right)$$

or

$$\arcsin\left(\frac{r}{r+h} \cos \epsilon\right) = \frac{\pi}{2} - \left(\epsilon + \frac{\varphi}{2n}\right)$$

Taking the sine of both sides gives

$$\epsilon = \arctan\left[\frac{\cos\frac{\varphi}{2n} - \frac{r}{r+h}}{\sin\frac{\varphi}{2n}}\right].$$

Further, for this case a simple application of the law of cosines gives

$$d = 2n\sqrt{r^2 + (r+h)^2 - 2r(r+h)\cos\frac{\varphi}{2n}}.$$

Unfortunately, in general (i.e., when $p \geqslant 2$) equation (1) cannot be solved for ϵ in closed form.

V. A Least Upper Bound for φ

Let a fixed ray configuration be considered. Clearly, $\epsilon = 0$ if and only if $\varphi = \max\varphi$. Substituting $\epsilon = 0$ in equation (1) gives

$$0 = \frac{1}{2}\left(\pi - \frac{\max\varphi}{N}\right) - \frac{1}{N}\sum_{i=1}^{p} n_i \arcsin K_i .$$

Hence,

$$\max\varphi = \sum_{i=1}^{p} n_i(\pi - 2\arcsin K_i).$$

If $\varphi > \max\varphi$, then the ray would have to pass through the earth; so if $0 < \varphi < \max\varphi$, then $0 < \epsilon < \pi/2$.

VI. Numerical Solution

Since equation (1) cannot, in general, be solved for ϵ in closed form, numerical methods are appealed to.

Set

$$g(\epsilon) = \frac{1}{2}\left(\pi - \frac{\varphi}{N}\right) - \frac{1}{N}\sum_{i=1}^{p} n_i \arcsin(K_i \cos\epsilon).$$

The following is a well known theorem: A sufficient condition for the convergence of the sequence of constants $\epsilon_0, \epsilon_1, \epsilon_2, \ldots$, which is given by the recursion formula

$$\epsilon_{n+1} = g(\epsilon_n), \quad n = 0, 1, 2, \ldots,$$

is that $g'(\epsilon)$ and a constant K exist such that

$$|g'(\epsilon)| \leqslant K < 1 \quad \text{holds for all } \epsilon.$$

(Here ϵ_0 is an arbitrary constant and $g(\epsilon)$ is assumed to be differentiable on the real line.)

Further, this sequence converges to the only real root of $\epsilon = g(\epsilon)$.

Now
$$g'(\epsilon) = \pm\frac{1}{N}\sum_{i=1}^{P} n_i K_i \sqrt{\frac{1-\cos^2\epsilon}{1-K_i^2\cos^2\epsilon}},$$

where
$$N = \sum_{i=1}^{P} n_i, \qquad K_i = \frac{r}{r+h_i}.$$

Since
$$0 \leqslant \sqrt{\frac{1-\cos^2\epsilon}{1-K_i^2\cos^2\epsilon}} \leqslant 1 \quad \text{on the real line,}$$

$$|g'(\epsilon)| \leqslant \frac{1}{N}\sum_{i=1}^{P} n_i K_i < 1 \quad \text{on the real line.}$$

In the iterative scheme given above any ϵ_0 may be used.

VII. Length of Ray Path

Applying the law of sines to the triangle OTP (Fig. 1) gives

$$\frac{d_i}{\sin\left[\frac{\pi}{2} - \left(\epsilon + \frac{\xi_i}{2}\right)\right]} = \frac{r+h_i}{\sin\left(\frac{\pi}{2}+\epsilon\right)},$$

where d_i is the length of side TP.

Using $\sin \xi_i/2 = K_i \cos\epsilon$,

$$d_i = \sqrt{2rh_i + h_i^2 + r^2\sin^2\epsilon} - r\sin\epsilon$$

is obtained.

Hence,

$$d = 2\sum_{i=1}^{P} n_i\sqrt{2rh_i + h_i^2 + r^2\sin^2\epsilon} - 2r\sin\epsilon\sum_{i=1}^{P} n_i,$$

where d is the total length that the ray travels in going from T to L.

VIII. Flat Earth Approximation

An estimate for ϵ may be obtained by assuming a flat earth and flat ionospheric layers. A short computation gives

$$\epsilon \approx \arc\tan\left[\frac{2}{r\varphi}\sum_{i=1}^{P} n_i h_i\right].$$

As φ goes to 0, this estimate will approach the value of ϵ computed by iteration.

IX. Earth – E_2 Layer – E_1 Layer – E_2 Layer – ··· – E_2 Layer – Earth Reflections

The ray configuration previously considered is now generalized, but for only 2 ionospheric layers E_1 and E_2 with $h_1 < h_2$.

The ray configuration, earth–E_2 layer–E_1 layer–E_2 layer–· · ·–E_2 layer–earth, with p reflections off the E_2 layer and $p - 1$ reflections off the E_1 layer will be called a hop of order p to the E_2 layer, $p \geqslant 2$.

The following ray configuration is now considered:

n hops to the E_1 layer,
m hops to the E_2 layer,
n_1 hops of order 2 to the E_2 layer,
n_2 hops of order 3 to the E_2 layer,
. . .
n_μ hops to order $\mu + 1$ to the E_2 layer.

X. Mathematical Model

Let ξ_1 be the angle made by the ray on reflecting off the E_1 layer.
Let ξ_2 be the angle made by the ray on reflecting off the E_2 layer.
Let ξ_3 be the angle made by the ray on reflecting off the E_1 layer in going from and to the E_2 layer (see Fig. 2).

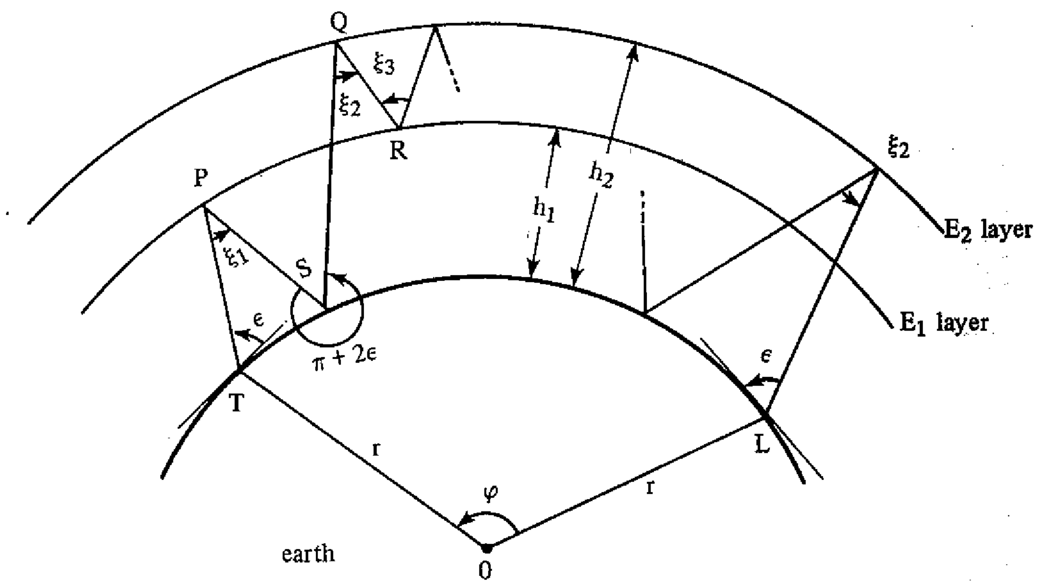

Fig. 2 — Earth – E_2 layer – E_1 layer – E_2 layer – ··· – E_2 layer – earth reflections.

By considering the polygon formed by the ray configuration and the radii of the earth (Fig. 2) and noting that the sum of the interior angles of this polygon is equal to $(\bar{n}-2)\pi$, where $\bar{n}$ is the number of vertices of the polygon,

$$\varphi + 2\left(\epsilon + \frac{\pi}{2}\right) + n\xi_1 + m\xi_2 + \left(\sum_{i=1}^{\mu}(i+1)n_i\right)\xi_2$$

$$+ \left(\sum_{i=1}^{\mu} in_i\right)(2\pi - \xi_3) + \left(n + m - 1 + \sum_{i=1}^{\mu} n_i\right)(\pi + 2\epsilon)$$

$$= \left(n + m + \sum_{i=1}^{\mu}(2i+1)n_i + n + m - 1 + \sum_{i=1}^{\mu} n_i + 3 - 2\right)\pi$$

is obtained.

Solving this equation for ϵ gives

$$\epsilon = \frac{1}{2}\left(\pi - \frac{\varphi}{n + m + \sum_{i=1}^{\mu} n_i}\right) - \frac{1}{2}(\lambda_1\xi_1 + \lambda_2\xi_2 - \lambda_3\xi_3),$$

where $$\lambda_1 = \frac{n}{n + m + \sum_{i=1}^{\mu} n_i}, \qquad \lambda_2 = \frac{m + \sum_{i=1}^{\mu}(i+1)n_i}{n + m + \sum_{i=1}^{\mu} n_i}, \qquad \lambda_3 = \frac{\sum_{i=1}^{\mu} in_i}{n + m + \sum_{i=1}^{\mu} n_i}.$$

Applying the law of sines to triangle OTP (see Fig. 2) gives

$$\xi_1 = 2 \arcsin (K_1 \cos \epsilon), \text{ where } K_1 = \frac{r}{r + h_1}.$$

A similar computation gives

$$\xi_2 = 2 \arcsin (K_2 \cos \epsilon) \text{ where } K_2 = \frac{r}{r + h_2}.$$

Applying the law of sines to triangle ORQ (see Fig. 2) gives

$$\frac{\sin\left(\pi - \frac{\xi_3}{2}\right)}{r + h_2} = \frac{\sin \frac{\xi_2}{2}}{r + h_1}.$$

Hence, $$\sin \frac{\xi_3}{2} = \frac{r + h_2}{r + h_1}\,\frac{r}{r + h_2} \cos \epsilon = K_1 \cos \epsilon$$

and so $\xi_1 = \xi_3$.

A short computation then gives

$$\varepsilon = \frac{1}{2}\left(\pi - \frac{\varphi}{n+m+\sum_{i=1}^{\mu} n_i}\right.$$

$$\left. - \left(\frac{n - \sum_{i=1}^{\mu} in_i}{n+m+\sum_{i=1}^{\mu} n_i} \arcsin(K_1 \cos\epsilon) + \frac{m + \sum_{i=1}^{\mu}(i+1)n_i}{n+m+\sum_{i=1}^{\mu} n_i} \arcsin(K_2 \cos\epsilon)\right)\right). \qquad (2)$$

This equation can be solved for ϵ in closed form only if $K_1 = K_2$ or if $n = \sum_{i=1}^{\mu} in_i$. The case $K_1 = K_2$ was treated in section IV. The case $n = \sum_{i=1}^{\mu} in_i$ can obviously be treated in the same way.

Set the right member of equation (2) to $g(\epsilon)$. Then

$$g'(\epsilon) = \pm \frac{n - \sum_{i=1}^{\mu} in_i}{n+m+\sum_{i=1}^{\mu} n_i} K_1 \sqrt{\frac{1-\cos^2\epsilon}{1-K_1^2\cos^2\epsilon}} \pm \frac{m + \sum_{i=1}^{\mu}(i+1)n_i}{n+m+\sum_{i=1}^{\mu} n_i} K_2 \sqrt{\frac{1-\cos^2\epsilon}{1-K_2^2\cos^2\epsilon}}.$$

Let $L_1(\epsilon) = \sqrt{\dfrac{1-\cos^2\epsilon}{1-K_1^2\cos^2\epsilon}}, \quad L_2(\epsilon) = \sqrt{\dfrac{1-\cos^2\epsilon}{1-K_2^2\cos^2\epsilon}}.$

Without loss of generality,

$$g'(\epsilon) = \frac{n - \sum_{i=1}^{\mu} in_i}{n+m+\sum_{i=1}^{\mu} n_i} K_1 L_1 + \frac{m + \sum_{i=1}^{\mu}(i+1)n_i}{n+m+\sum_{i=1}^{\mu} n_i} K_2 L_2$$

(otherwise consider $-g'(\epsilon)$). Now $0 < K_2 < K_1 < 1$, $0 \leqslant L_2 \leqslant L_1 \leqslant 1$, so $0 \leqslant K_2L_2 \leqslant K_1L_1$ and

$$0 \leqslant \frac{m + \sum_{i=1}^{\mu}(i+1)n_i}{n+m+\sum_{i=1}^{\mu} n_i} K_2 L_2 \leqslant \frac{m + \sum_{i=1}^{\mu}(i+1)n_i}{n+m+\sum_{i=1}^{\mu} n_i} K_1 L_1.$$

Adding $\dfrac{n - \sum_{i=1}^{\mu} in_i}{n+m+\sum_{i=1}^{\mu} n_i} K_1 L_1$ to each member of this double inequality gives

$$\frac{n - \sum_{i=1}^{\mu} i n_i}{n + m + \sum_{i=1}^{\mu} n_i} K_1 L_1 \leqslant g'(\epsilon) \leqslant K_1 L_1 \leqslant K_1 < 1.$$

If $n - \sum_{i=1}^{\mu} i n_i \geqslant 0$, then clearly $0 \leqslant g'(\epsilon) \leqslant K_1 < 1$. Further, a short computation will verify that if

$$n - \sum_{i=1}^{\mu} i n_i < 0, \text{ then } |g'(\epsilon)| \leqslant K_1 L_1 \leqslant K_1 < 1 \text{ if and only if } \sum_{i=1}^{\mu} i n_i \leqslant 2n + m + \sum_{i=1}^{\mu} n_i.$$

Note that $\sum_{i=1}^{\mu} i n_i > 2n + m + \sum_{i=1}^{\mu} n_i$ implies $\sum_{i=1}^{\mu} i n_i > n$, i.e., $n - \sum_{i=1}^{\mu} i n_i < 0$. Unfortunately, it is *not* true that $|g'(\epsilon)| \leqslant K_1 < 1$ for this case. However, there is some consolation in that the case $n = m = 0$, $n_1 = 1, n_2 = \cdots = n_\mu = 0$, which is of prime interest, does satisfy $\sum_{i=1}^{\mu} i n_i \leqslant 2n + m + \sum_{i=1}^{\mu} n_i$.

Hence if $n - \sum_{i=1}^{\mu} i n_i \geqslant 0$ or if $n - \sum_{i=1}^{\mu} i n_i < 0$ and $\sum_{i=1}^{\mu} i n_i \leqslant 2n + m + \sum_{i=1}^{\mu} n_i$, then the numerical solution of equation (2) is the same as that for equation (1). In fact (for the case $p = 2$) equation (1) can be obtained from equation (2) by letting $n_1 = n_2 = \cdots = n_\mu = 0$.

As before

$$\max \varphi = \left(n + m + \sum_{i=1}^{\mu} n_i\right)\pi - 2\left[\left(n - \sum_{i=1}^{\mu} i n_i\right) \arcsin K_1 + \left(m + \sum_{i=1}^{\mu} (i+1) n_i\right) \arcsin K_2\right].$$

Let d_1 be the length of TP, d_2 be the length of SQ, and d_3 be the length of RQ (see Fig. 2).

A short computation (see section VII) gives

$$d_1 = \sqrt{2rh_1 + h_1^2 + r^2 \sin^2 \epsilon} - r \sin \epsilon,$$
$$d_2 = \sqrt{2rh_2 + h_2^2 + r^2 \sin^2 \epsilon} - r \sin \epsilon.$$

Applying the law of sines to triangle OQR (see Fig. 2) gives

$$\frac{d_3}{\sin\left(\frac{\xi_3}{2} - \frac{\xi_2}{2}\right)} = \frac{r + h_2}{\sin\left(\pi - \frac{\xi_3}{2}\right)} = \frac{r + h_2}{\sin \frac{\xi_3}{2}}$$

But
$$\sin \frac{\xi_3}{2} = \frac{r}{r + h_1} \cos \epsilon, \quad \sin \frac{\xi_2}{2} = \frac{r}{r + h_2} \cos \epsilon.$$

Combining these three equations gives

$$d_3 = (r + h_2)\sqrt{1 - K_2^2 \cos^2 \epsilon} - (r + h_1)\sqrt{1 - K_1^2 \cos^2 \epsilon}.$$

And so

$$d = 2\left[nd_1 + \left(m + \sum_{i=1}^{\mu} n_i\right)d_2 + \left(\sum_{i=1}^{\mu} in_i\right)d_3\right],$$

where d is the total length that the ray travels in going from T to L.

The flat earth approximation for ϵ is given by

$$\arctan\left\{\frac{2}{r\varphi}\left[\left(n - \sum_{i=1}^{\mu} in_i\right)h_1 + \left(m + \sum_{i=1}^{\mu} (i+1)n_i\right)h_2\right]\right\}.$$